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A conjecture on the stability and mixing of non-parallel shear flows

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Non-parallel shear flows of an inviscid, incompressible, density-stratified fluid are considered. The stability is studied in terms of the possibility of complete mixing within a horizontal layer of given thickness. It is assumed that the energy interchange between the mixed region and the external fluid can be neglected. It is also assumed that the required turbulent energy is greater than the energy needed to invert the region to be mixed. The term 'stable' as used here means that the kinetic energy released by making the velocity constant over the layer thickness is not sufficient to provide the required turbulent energy for that layer thickness.

If each horizontal fluid plane has a translational velocity of the same magnitude and shear is produced by rotation of the velocity vector with increasing height, then 'stability' increases strongly with increasing layer thickness.

1. Introduction

Miles (1961) established a sufficient condition for the dynamic stability of a heterogeneous parallel shear flow subjected to infinitesimal disturbances. This condition is that the local Richardson number J is everywhere greater than $\frac{1}{4}$ and $U'(z) \neq 0$. ($J = N^2/U'^2$, where N is the Brunt–Väisälä frequency and U' is the vertical gradient of the horizontal velocity.) This confirmed an earlier conjecture by Taylor (1931). Howard (1961) presented a simplified proof requiring fewer assumptions. References to other work in this field can be found in Miles (1961) and Yih (1965, 1974).

Here we treat, by a less precise method, a *non*-parallel shear flow with constant local density gradient. The shear is produced by maintaining a fixed resultant horizontal velocity, but rotating the direction of the resultant velocity vector uniformly with increasing height.

Instead of specifying small disturbances, we proceed to the ultimate result of instability and ask if a horizontal layer of fluid of given thickness can release sufficient energy from its non-uniform velocity profile to provide the stipulated minimum of turbulent energy. We assume that during the transition to turbulence there is no significant energy exchange between the mixing region and the external fluid. The minimum energy required for turbulence is taken as that required to overturn or invert the layer (which initially has heavier fluid particles at the bottom). The mechanism of turbulence is of course much more complicated than this would suggest. Turbulence involves motions in three dimensions and a flow of energy from larger-scale to smaller-scale motions. Nevertheless the *least* energy that such motion could possess (immediately after transition) can plausibly be set as that required to invert the layer, if complete mixing is to be attained. Although computed from a potential-energy requirement this energy would appear in the mixed region as both kinetic and potential turbulent energy.

The assumption of zero energy exchange between the mixing region and the external fluid is not as arbitrary as one might at first suppose. Above and below the turbulent mixed region unsteady motions appear, but decrease rapidly with increasing vertical distance. These non-turbulent regions resemble the turbulent region in two important respects. Energy is liberated if the velocity gradients are reduced and energy (potential and kinetic) is required to maintain the unsteady motions. Beyond the regions of large unsteady motion are regions of small disturbance across whose boundaries no significant amount of energy can flow (since pressure and velocity perturbations are small). Thus the present analysis might be said to deal with layers of large disturbance (containing turbulent and non-turbulent flow) whose characteristics are approximated by complete mixing.

In considering the assumption of zero energy exchange it should also be noted that the sufficient condition for stability of a fluid subjected to small disturbances does not necessarily depend on the extent of the fluid. The condition applies to a thin horizontal layer of fluid bounded by solid walls, and precisely the same condition applies to this thin layer when it is embedded in an infinite expanse of stable fluid. This suggests the unimportance of energy exchange between the local unstable region and the external fluid.

2. Development

The x and y axes are horizontal and the z axis (or kz axis) is vertical (see figure 1). The density of the incompressible fluid is taken as $\rho(z) = \rho_0 + \rho' z$ with ρ' constant. (Here the subscript 0 denotes the value at $z = 0$ and the prime indicates differentiation with respect to z.) The magnitude of the translational velocity U of each horizontal fluid plane is assumed constant, but the direction varies with height, such that $u = U \sin(kz)$ and $v = U\cos(kz)$, u and v being the x and y components of U. Then $u'^2 + v'^2 = U^2k^2$ and $J = N^2/U^2k^2$, where $N^2 = -g\rho'/\rho_0$.

Initially the mass, x momentum, y momentum and kinetic energy, each per unit of horizontal area, are

$$\left.\begin{aligned} \text{mass} &= \int_0^h \rho\, dz, \qquad x\,\text{momentum} = \int_0^h \rho u\, dz, \\ y\,\text{momentum} &= \int_0^h \rho v\, dz, \qquad \text{k.e.} = \frac{1}{2}\int_0^h \rho U^2 dz, \end{aligned}\right\} \tag{1}$$

and the potential energy (which contains an arbitrary reference level) is set equal to zero.

After mixing, these quantities are

$$\left.\begin{aligned} \text{mass} &= (\rho_0 + \Delta\rho)\, h, \\ x\,\text{momentum} &= (\rho_0 + \Delta\rho)(u_0 + \Delta u)\, h, \\ y\,\text{momentum} &= (\rho_0 + \Delta\rho)(v_0 + \Delta v)\, h, \\ \text{k.e.} &= \tfrac{1}{2}(\rho_0 + \Delta\rho)\,[(u_0 + \Delta u)^2 + (v_0 + \Delta v)^2]\, h \end{aligned}\right\} \tag{2}$$

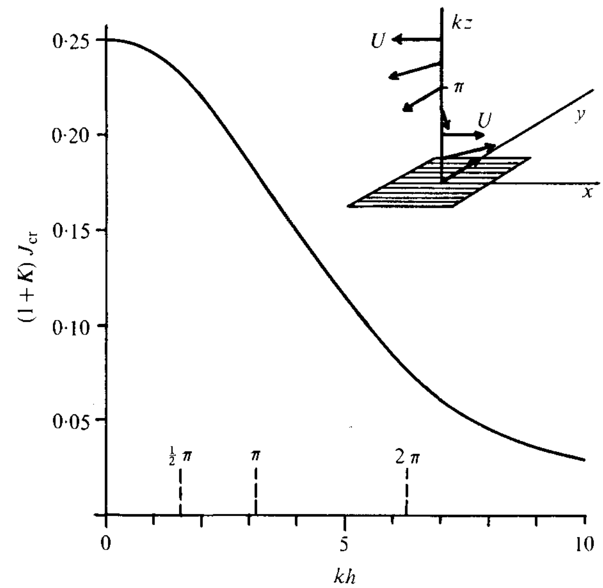

FIGURE 1. Critical Richardson number for 'stability' *vs.* non-dimensional layer thickness.

and the turbulent energy is expressed as the increase in potential energy produced by inverting the layer multiplied by $1+K$, where K is a positive constant:

$$\text{t.e.} = (1+K)g\int_0^h (\rho_0+\rho' z)[(h-z)-z]\,dz = -\tfrac{1}{6}g\rho' h^3(1+K). \tag{3}$$

(The buoyant force does no net work in this exchange of particles.)

The quantities $\Delta\rho$, Δu and Δv are determined by the conservation of mass, x momentum and y momentum and become

$$\Delta\rho = \tfrac{1}{2}\rho' h, \tag{4}$$

$$\Delta u = U[1-\cos(kh)]/kh, \tag{5}$$

$$\Delta v = U[\sin(kh)-kh]/kh. \tag{6}$$

The 'stability' boundary comes from equating the total energies before and after mixing, which gives

$$\tfrac{1}{2}(\rho_0+\Delta\rho)[(u_0+\Delta u)^2+(v_0+\Delta v)^2]h - \tfrac{1}{2}U^2\int_0^h(\rho_0+\rho' z)\,dz - \tfrac{1}{6}g\rho' h^3(1+K) = 0. \tag{7}$$

In the first two terms (the kinetic-energy terms) $\rho' h$ can be neglected in comparison with ρ_0. This is a Boussinesq approximation, the effect of density variation on inertia terms being much less than its effect on restoring forces. Then $\Delta\rho$ in the first term in (7) and ρ' in the second term can be set equal to zero. Δu and Δv can be replaced from (5) and (6), and noting that $u_0 = 0$, $v_0 = U$ and $-g\rho'/\rho_0 U^2k^2 = J$ we get

$$J_{cr} = 3[k^2h^2+2\cos kh-2]/k^4h^4(1+K). \tag{8}$$

Here J_{cr} is the critical value of the Richardson number corresponding to the 'stability' boundary. This relation is plotted in figure 1. Above the boundary the fluid is 'stable' (i.e. cannot mix completely). Below the boundary the fluid is 'unstable' (i.e. can release enough energy from its velocity profile to overturn the layer and mix completely).

3. Discussion and conclusions

When h, the layer thickness, approaches zero (8) can be evaluated by expanding the cosine term and gives

$$J_{cr} = [4(1+K)]^{-1}, \tag{9}$$

where K is always greater than zero. When hk approaches ∞, (8) approaches

$$J_{cr} = 3[k^2h^2(1+K)]^{-1}. \tag{10}$$

This indicates that a layer of large thickness (in terms of the vertical period $2\pi/k$) does not readily mix completely and uniformly over its entire thickness.

Equation (10) also applies (precisely) when kh is an even multiple of π. If $kh = 2\pi$, $J_{cr} = 3[4\pi^2(1+K)]^{-1}$ and, even if K is assigned the minimum value of zero, $J_{cr} = 0{\cdot}076$, which is considerably less than 0·25, the value for parallel shear subjected to small perturbations. This means that a layer of thickness $2\pi/k$ cannot mix completely and uniformly for a Richardson number greater than 0·076.

In weighing up such results one must keep in mind the initial assumptions concerning the absence of significant energy exchange with the external fluid during transition and the prescription of a minimum turbulent energy consistent with complete mixing. Also one should consider that K, here assumed constant, might perhaps vary with the layer thickness.

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